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Cancellation exponent and multifractal structure in two-dimensional magnetohydrodynamics: Direct numerical simulations and Lagrangian averaged modeling

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We present direct numerical simulations and Lagrangian averaged (also known as α model) simulations of forced and free decaying magnetohydrodynamic turbulence in two dimensions. The statistics of sign cancellations of the current at small scales is studied using both the cancellation exponent and the fractal dimension of the structures. The α model is found to have the same scaling behavior between positive and negative contributions as the direct numerical simulations. The α model is also able to reproduce the time evolution of these quantities in free decaying turbulence. At large Reynolds numbers, an independence of the cancellation exponent with the Reynolds numbers is observed.

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The magnetohydrodynamic (MHD) approximation is often used to model plasmas or conducting fluids in astrophysical and geophysical environments. However, given the huge amount of temporal and spatial scales involved in the dynamics of these objects, simulations are always carried out in a region of parameter space far from the observed values. Lagrangian averaged magnetohydrodynamics (LAMHD), also called the MHD alpha model [1,2] (or the Camassa-Holm equations in early papers studying the hydrodynamic case [3]), has been recently introduced as a way to reduce the number of degrees of freedom of the system, while keeping accurate evolution for the large scales. This approach {as well as large eddy simulations (LES) for MHD; see, e.g., Ref. [4]} is intended to model astrophysical or geophysical flows at high Reynolds numbers using available computational resources. Several aspects of the MHD alpha model have already been tested in two and three dimensions at moderate Reynolds numbers, against direct numerical simulations of the MHD equations [2]. These studies were focused on comparisons of the evolution of global quantities and the dynamics of the large scale components of the energy spectrum [2,5].

All these models introduce changes in the small scales in order to preserve the evolution of the large scales. In several cases, it is of interest to know the statistics of the small scales. It is also important to model properly the small scales because they have an effect on large scales, as for example in the case of eddy noise: the beating of two small scales eddies produces energy at the large scale, and this may affect the global long-time evolution of the flow, an issue that arises in global climate evolution or in solar-terrestrial interactions. Moreover, plasmas and conducting fluids generate thin and intense current sheets where magnetic reconnection takes place. In these regions, the magnetic field and the current rapidly change sign, and after reconnection the magnetic energy is turned into mechanical and thermal energy. These events are known to take place in the magnetopause [6], the magnetotail [7], the solar atmosphere [8], and the interplanetary medium [9].

Current sheets are strongly localized and intermittent. To preserve reliable statistics of these events in models of MHD turbulence is of utmost importance to model some of these astrophysical and geophysical problems. In this work, we study whether the MHD alpha model is able to reproduce the statistics and scaling observed in these phenomena.

In order to measure fast oscillations in sign of a field on arbitrary small scales, the cancellation exponent was introduced [10–12]. The exponent is a measure of sign singularity. We can define the signed measure for the current $j_z(\mathbf{x})$ on a set $Q(L)$ of size L as

$$\mu_i(l) = \int_{Q_i(l)} d\mathbf{x}\, j_z(\mathbf{x}) \Big/ \int_{Q(L)} d\mathbf{x} |j_z(\mathbf{x})|, \tag{1}$$

where $\{Q_i(l)\} \subset Q(L)$ is a hierarchy of disjoint subsets of size l covering $Q(L)$. The partition function χ measures the cancellations at a given lengthscale l,

$$\chi(l) = \sum_{Q_i(l)} |\mu_i(l)|. \tag{2}$$

Note that for noninteger L/l the subsets will not cover $Q(L)$ and finite size box effects must be considered in the normalization of Eq. (1). We can study the scaling behaviors of the cancellations defining the cancellation exponent κ, where

$$\chi(l) \sim l^{-\kappa}. \tag{3}$$

Positive κ indicates fast changes in sign on small scales (in practice, a cutoff is always present at the dissipation scale). A totally smooth field has $\kappa=0$. This exponent can also be related with the fractal dimension D of the structures [12],

$$\kappa = (d - D)/2, \tag{4}$$

where d is the number of spatial dimensions of the system. In some circumstances, we will also be interested on the cancellation exponent for the vorticity ω_z. In that case the vorticity replaces the current in the definition of $\mu_i(l)$ [Eq. (1)].

Under special assumptions, relations between the cancellation exponent and scaling exponents have also been derived [11]. Positive cancellation exponent κ has been found in plasma experiments [10], direct simulations of MHD turbulence [12], *in situ* solar wind observations [13], and solar photospheric active regions [14], where changes in the scaling were identified as preludes to flares.

In this work we will consider both free decaying and forced simulations of incompressible MHD and LAMHD turbulence in two dimensions (2D). The MHD equations in 2D can be written in terms of the stream function Ψ and the z component of the vector potential A_z,

$$\partial_t \nabla^2 \Psi = [\Psi, \nabla^2 \Psi] - [A_z, \nabla^2 A_z] + \nu \nabla^4 \Psi \tag{5}$$

$$\partial_t A_z = [\Psi, A_z] + \eta \nabla^2 A_z, \tag{6}$$

where the velocity and magnetic field are given by $\mathbf{v} = \nabla \times (\Psi \hat{z})$ and $\mathbf{B} = \nabla \times (A_z \hat{z})$ respectively, and $[F,G] = \partial_x F \partial_y G - \partial_x G \partial_y F$ is the standard Poisson bracket. The LAMHD equations are obtained by introducing a smoothing length α, and the relation between smoothed (denoted by a subindex s) and unsmoothed fields is given by $\mathbf{F} = (1 - \alpha^2 \nabla^2) \mathbf{F}_s$, for any field $\mathbf{F}$. The system of LAMHD equations in this geometry [2] is

$$\partial_t \nabla^2 \Psi = [\Psi_s, \nabla^2 \Psi] - [A_{s_z}, \nabla^2 A_z] + \nu \nabla^4 \Psi \tag{7}$$

$$\partial_t A_{s_z} = [\Psi_s, A_{s_z}] + \eta \nabla^2 A_z. \tag{8}$$

For both systems of equations, the current is given by $j_z = -\nabla^2 A_z$, and the vorticity by $\omega_z = -\nabla^2 \Psi$. In these equations and in all the following figures, all quantities will be given in familiar Alfvénic dimensionless units. Equations (5)–(8) are solved in a periodic box using a pseudospectral code as described in Ref. [2]. The code implements the 2/3 rule for dealiasing, and the maximum wave number resolved is $k_{max} = N/3$, where N is the linear resolution used in the simulation. All the fields are written in dimensionless units.

To characterize the oscillating behavior and sign singularities in the flows obtained from the MHD and LAMHD simulations, we perform a signed measure analysis and compute the cancellation exponent κ for the current and for the vorticity. Following Eq. (3), its value is obtained by fitting $\chi(l) = c(l/L)^{-\kappa}$ through the inertial range, where $L = 2\pi$ is the length of the box, and c is a constant. The length scales in the inertial range used for this fit are obtained studying the scaling of the third order structure function [15].

We first present results for a forced MHD simulation with 1024^2 grid points, with $\eta = \nu = 1.6 \times 10^{-4}$. Both the momentum and the vector potential equations were forced. The external forces had random phases in the Fourier ring between $k=1$ and $k=2$, and a correlation time of $\Delta t = 5 \times 10^{-2}$. The system was evolved in time until reaching a turbulent steady state. The amplitude of the magnetic force averaged over space was held constant to 0.2, and the amplitude of the mechanical force to 0.45, in order to have the system close to equipartition. Two more simulations using the LAMHD system were carried out, with the same parameters as the MHD run but with resolutions of 512^2 grid points ($\alpha \approx 0.0117$), and 256^2 grid points ($\alpha \approx 0.0234$), respectively, (the choice $\alpha = 2/k_{max}$ is conventional [2,3]). The Kolmogorov's kinetic and magnetic dissipation wave numbers in the MHD run are $k_\nu \approx k_\eta \approx 332$; in all the LAMHD simulations these wave numbers are larger than the largest resolved wave number k_{max}, by virtue of the model. Note that although it is common to reduce the spatial resolution even more in studies of the large scale components of the energy spectrum in LES of hydrodynamic turbulence, this cannot be done in this context since wide energy spectra and large amounts of spatial statistics are needed to properly compute the cancellation exponent (see, e.g., Ref. [16] for a study of intermittency in LES).

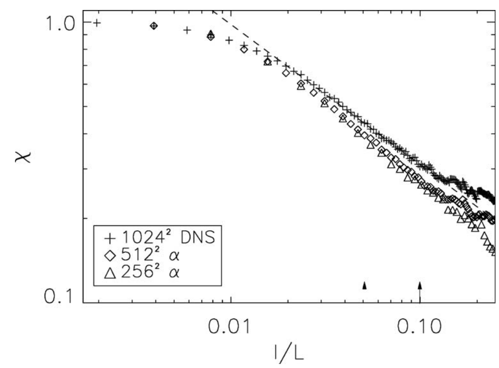

FIG. 1. $\chi(l)$ averaged in time for j_z in forced MHD turbulence. The pluses correspond to the 1024^2 MHD simulation, diamonds to the 512^2 LAMHD run, and triangles to the 256^2 LAMHD run. The dashed line indicates a slope of 0.50. The arrows indicate the inertial range. Note that the slopes are of import, not the offsets.

Figure 1 shows $\chi(l)$ for the three simulations, averaged using 11 snapshots of the current covering a total time span of 20 turnover times in the turbulent steady state. A power law can be identified at intermediate scales, scales smaller than the forcing band but larger than the dissipation scale. Note that the two LAMHD simulations reproduce the same scaling as the MHD simulation. As a result, the sign singularity and fractal structure are both well captured in the inertial range although the alpha model is known to give thicker structures at scales smaller than α due to the introduction of the smoothing length [2,17]. The best fit for the current j_z using a power law in the inertial range gives $\kappa = 0.50 \pm 0.17$ for the 1024^2 MHD run, $\kappa = 0.55 \pm 0.19$ for the 512^2 LAMHD simulation, and $\kappa = 0.55 \pm 0.43$ for the 256^2 LAMHD simulation. Note that a value of $\kappa = 0.50$ in the MHD simulation gives a value of the fractal dimension $D = 1.00 \pm 0.34$, close to the codimension of 1 corresponding to current sheets in MHD turbulence. For the vorticity, the cancellation exponent is $\kappa = 0.73 \pm 0.16$ for the 1024^2 MHD run, $\kappa = 0.74 \pm 0.32$ for the 512^2 LAMHD simulation, and $\kappa = 0.80 \pm 0.32$ for the 256^2 LAMHD simulation, giving a fractal dimension of $D = 0.54$ in the MHD simulation. The values obtained are compatible with the values of $\kappa = 0.43 \pm 0.06$ and $D = 1.14 \pm 0.12$ for the current, and $\kappa = 0.69 \pm 0.12$ and $D = 0.62 \pm 0.24$ for the vorticity obtained in Ref. [12] for forced direct numerical simulations of 2D MHD turbulence using a 1024^2 spatial grid and $\eta = \nu = 8 \times 10^{-4}$. Given the good agreement between MHD and LAMHD simulations, in the following we will only refer to the cancellation exponent for the current density.

Figure 2(a) shows the corresponding results for free decaying MHD turbulence. Three simulations are shown, one MHD run using 2048^2 grid points, a 1024^2 LAMHD run with $\alpha \approx 0.0058$, and a 512^2 LAMHD run with $\alpha \approx 0.0117$. The

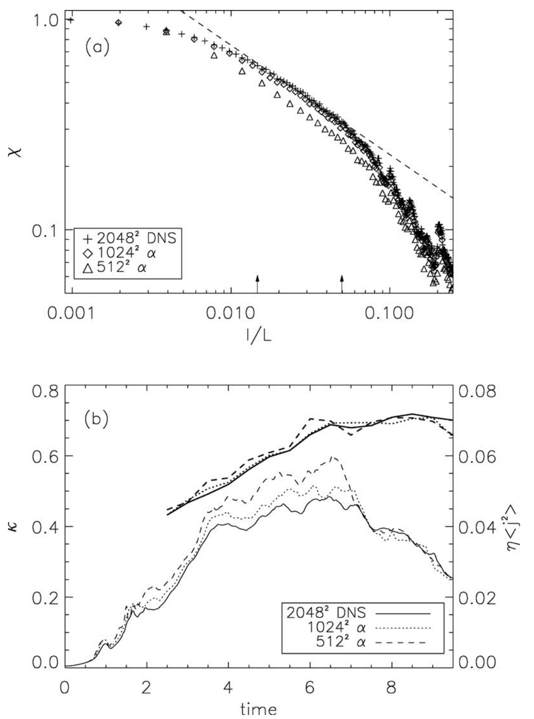

FIG. 2. (a) $\chi(l)$ at $t=4$ in the free decaying simulations, pluses correspond to the 2048^2 MHD simulation, diamonds to the 1024^2 LAMHD run, and triangles to the 512^2 LAMHD run (the dashed line indicates a slope of 0.52 and the arrows indicate the inertial range); (b) time history of the cancellation exponent (thick lines) for the three runs, and of $\eta\langle j_z^2\rangle$, where the brackets denote spatial average.

three simulations were started with the same initial conditions; initial velocity and magnetic fields with random phases between $k=1$ and $k=3$ in Fourier space, and unit rms values. The kinematic viscosity and magnetic diffusivity used were $\nu=\eta=10^{-4}$. The three simulations were evolved in time without external forces.

The evolution of the cancellation exponent as a function of time in the free decaying simulations is shown in Fig. 2(b). For these simulations, the cancellation exponent is computed between the lengthscales $L/l\approx 20$ and $L/l\approx 70$, where a power law scaling in $\chi(l)$ can be clearly identified from $t=2.5$ up to $t=10$. At $t=0$ the cancellation exponent κ is zero, which corresponds to the smooth initial conditions. A gap between $t=0$ and $t=2.5$ is present where no clear scaling is observed. As time evolves, κ grows up to 0.75 at $t\approx 8$, as the system evolves from the initially smooth fields to a turbulent state with strong and localized current sheets. After this maximum, the exponent κ decays slowly in time. The maximum of κ takes place slightly later than the maximum of magnetic dissipation, as is also shown in Fig. 2(b). Note that the alpha model also captures the time evolution of the cancellation exponent in free decaying turbulence, as well as the fractal structure of the problem as time evolves.

As previously noted in Ref. [2], the alpha model slightly overestimates the magnetic dissipation. Note, however, that in the three simulations the peak of magnetic dissipation takes place close to $t\approx 6$, just before the peak of the cancellation exponent κ. From the maximum energy dissipation rate, the Kolmogorov's dissipation wave number for the kinetic and magnetic energy at $t\approx 6$ are estimated as $k_\nu\approx k_\eta\approx 470$, and this is again larger than the largest wavenumbers resolved in the two LAMHD simulations.

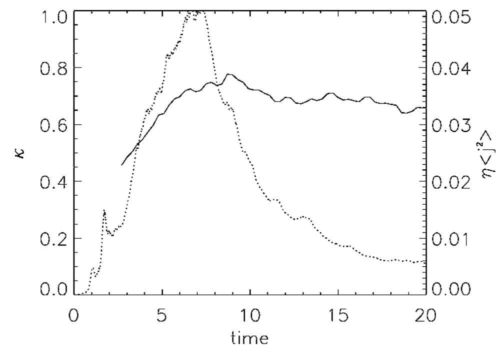

FIG. 3. Time history of κ (solid line) and $\eta\langle j_z^2\rangle$ (dotted line), for a free decaying LAMHD simulation with $\eta=\nu=2\times 10^{-5}$.

The observed slow decay of the cancellation exponent (compared with the square current) is related to the persistence of strong current sheets in the system for long times, even after the peak of magnetic dissipation. The system, instead of evolving fast to a smooth solution at every point in space, keeps dissipating energy in a few thin localized structures. The existence of these current sheets at late times can be more easily verified in simulations with smaller viscosity ν and diffusivity η. While in the peak of magnetic dissipation the system is permeated by a large number of small current sheets, at late times only a few current sheets are observed isolated by large regions where the fields are smooth.

Given the good agreement between direct numerical simulations (DNS) and LAMHD as seen in the preceding figure, we can reliably explore with the model Reynolds numbers unattainable in a reasonable time with DNS. In this context, we show that the maximum values of κ obtained in the simulations seem to be insensitive to the Reynolds numbers within a given method (MHD or LAMHD) once a turbulent state is reached. As an example, in Fig. 3 we give the time history of the cancellation exponent and the square current for a free decaying LAMHD simulation with $\eta=\nu=2\times 10^{-5}$ up to $t=20$. The initial conditions are the same as in the previously discussed simulations, and $\alpha\approx 0.0033$. It is worth noting that the time evolution of the magnetic dissipation in both decaying runs [Figs. 2(b) and 3] confirm previous results at lower Reynolds numbers [18,19]: namely that the peak dissipation ($t\sim 7$) is lower for higher Reynolds numbers, while for later times it is quite independent of the Reynolds values.

Figure 4 shows $\chi(l)$ for early and late times in the same simulation. At small scales, the slope of χ always goes to zero, as can be expected since close to the dissipation lengthscale the fields are expected to be smooth. However, note that as time evolves the scaling of χ with l drifts to smaller scales, and at $t=20$ a scaling can be observed up to $l/L\approx 0.005$. By virtue of the model the scaling is wider and the slope goes to zero faster than in the DNS due to the larger Reynolds number.

The statistics of sign cancellation in magnetofluid turbulence are related with intermittency and anomalous scaling

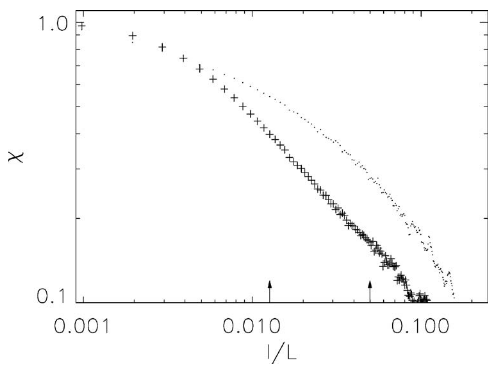

FIG. 4. $\chi(l)$ at $t=3$ (dots), and $t=20$ (pluses), for the free decaying LAMHD simulation with $\eta=\nu=2\times10^{-5}$. The arrows indicate the inertial range.

of structure functions [11], inherently associated with the dynamics of the small scales, and as a result harder to model in truncations or closures of the MHD equations. For example, two-point closures of turbulence behave smoothly (they also have no information about physical structures since they deal only with energy spectra). The intermittency of LES is an open topic, in particular because for neutral fluids the need to study the three dimensional case implied until recently that only low-resolution studies could be accomplished for which intermittent structures were barely resolved (see Ref. [20] for a recent study). From that point of view, the present study in two dimensions allows for higher Reynolds number studies. In MHD turbulence, the energy cascade being to small scales both in two and three space dimensions, it is hoped that the information gained here will carry on to the three-dimensional case. The result stemming from this study is that the LAMHD alpha model, although it alters the small scales through filtering, it nevertheless preserves some statistical information concerning the small scales. It is able to reproduce the scaling observed in forced MHD turbulence, as well as the time evolution of the cancellation exponent in free decaying simulations and as such, it represents a valuable model for studies of MHD flows for example at low magnetic Prandtl number ν/η as encountered in the liquid core of the Earth or in the solar convection zone (see, e.g., Ref. [5]).

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